

SOME REMARKS ON CONES OF PARTIALLY AMPLE DIVISORS

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ABSTRACT. We study the cones of q -ample divisors $q\text{Amp}$ on smooth complex varieties. In favourable cases, we identify a part where the closure $\overline{q\text{Amp}}$ and the nef cone have the same boundary. This is especially interesting for Fano (or almost Fano) varieties.

Totaro’s landmark paper [25] has given a new impetus to the study of partially ample divisors. Let X be a smooth projective complex variety of dimension n , and L on X a line bundle. We recall that L is called q -ample if for every coherent sheaf \mathcal{F} there exists an integer m_0 such that

$$H^i(X, \mathcal{F} \otimes L^{\otimes m}) = 0 \text{ for all } i > q \text{ and } m > m_0.$$

From Serre’s criterion it follows that 0-ampleness coincides with ampleness. Totaro proves that the q -ampleness of L only depends on the numerical equivalence class of L [25, Theorem 8.3]. The definition can moreover be extended to \mathbb{R} -divisors [25, 8.2], in such a way that q -ample \mathbb{R} -divisors form an open cone $q\text{Amp}(X)$ in $N^1(X)$ (the space of \mathbb{R} -divisors modulo numerical equivalence). We thus get a series of cones

$$\text{Amp}(X) = 0\text{Amp}(X) \subset 1\text{Amp}(X) \subset \cdots \subset n\text{Amp}(X) = N^1(X).$$

While the ample cone $\text{Amp}(X)$ and the cone $(n-1)\text{Amp}(X)$ are fairly well understood, the intermediate cones $q\text{Amp}(X)$ for $0 < q < n-1$ are still quite elusive and mysterious (see for instance [25, section 11] for some fundamental open questions).

The modest goal of this paper is to identify a part of these cones $q\text{Amp}$. Indeed, it turns out that in favourable cases, part of the boundary of the closed cone $\overline{q\text{Amp}}$ coincides with the boundary of the nef cone. To start with, let’s restrict attention to the case that is easiest to state, that of the cone of 1-ample divisors 1Amp . Let $\partial\text{Nef}(X)$ denote the boundary of the nef cone, and let $K_X \in N^1(X)$ denote the class of the canonical divisor. We define

$$\partial\text{Nef}(X)_{\text{visible}} \subset \partial\text{Nef}(X)$$

to be the part of the boundary that is visible from K_X ; cf. Definition 17 for the precise definition. (We note that when K_X is nef, we have $\partial\text{Nef}(X)_{\text{visible}} = \emptyset$!)

This “ K_X -visible part” of the boundary turns out to be closely related to the boundary of $1\text{Amp}(X)$. This is detailed in the following result, where $\text{Mob}(X)$ and $\text{Big}(X)$ denote the cone of mobile divisors resp. big divisors.

Theorem. (=Theorem 19) *Let X be a smooth projective complex variety.*

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(i)

$$\partial \text{Nef}(X)_{\text{visible}} \cap \text{int}(\text{Mob}(X))$$

is in the boundary of $\overline{1\text{Amp}(X)}$.

(ii) Suppose X is not the blow-up of a smooth projective variety along a smooth codimension 2 subvariety. Then

$$\partial \text{Nef}(X)_{\text{visible}} \cap \text{Big}(X) \subset \overline{\partial 1\text{Amp}(X)}.$$

(iii) Suppose X is not a conic bundle over a smooth projective variety, nor a blow-up of a smooth projective variety along a smooth codimension 2 subvariety. Then

$$\partial \text{Nef}(X)_{\text{visible}} \subset \overline{\partial 1\text{Amp}(X)}.$$

That is, with two exceptions (a blow-up and a conic bundle) the ample cone and the 1-ample cone look exactly the same when observed from K_X , and hence the only places where 1Amp can grow larger than Amp are located in the “shadowy part” invisible from K_X . This theorem is proven by exploiting the existence of an MMP for any adjoint divisor, as proven by Birkar–Cascini–Hacon–McKernan [5].

It follows from Theorem 19 that the cone $1\text{Amp}(X)$ is strictly convex for any X such that $\partial \text{Nef}(X)_{\text{visible}} \cap \text{int}(\text{Mob}(X)) \neq \emptyset$ (Corollary 24). The following is also an immediate corollary:

Corollary. (=Corollary 23) *Let X be a smooth projective variety, and suppose K_X is 1-ample. Then*

$$\partial \text{Nef}(X)_{\text{visible}} \subset \overline{\partial \text{Mob}(X)}.$$

That is, if K_X is 1-ample the nef cone and the closed mobile cone look the same when observed from K_X .

Of course, the above theorem is empty of content when K_X is nef (for then the K_X -visible part is empty), while the assertion grows stronger when K_X grows more negative (for then the K_X -visible part grows larger, which means that the 1-ample cone looks more and more like the ample cone). The limit case is when X is a Fano variety: then the whole boundary of $\text{Nef}(X)$ is K_X -visible. In fact, we can prove more generally:

Corollary. (=Corollary 25) *Let X be a smooth projective complex variety such that either (1) $-K_X$ is ample, or (2) $-K_X$ is $\neq 0$ and nef and $\dim N^1 X \geq 3$. Then:*

(i)

$$\partial \text{Nef}(X) \cap \text{int}(\text{Mob}(X))$$

is in the boundary of $\overline{1\text{Amp}(X)}$.

(ii) Suppose X is not the blow-up of a smooth projective variety Y along a smooth codimension 2 subvariety. Then

$$\partial \text{Nef}(X) \cap \text{Big}(X) \subset \overline{\partial 1\text{Amp}(X)}.$$

(iii) Suppose X is not a conic bundle over a smooth projective variety Y , nor a blow-up of a smooth projective variety along a smooth codimension 2 subvariety. Then

$$\text{Amp}(X) = 1\text{Amp}(X).$$

(For Fano varieties, I proved this in [21]).

Here is an application of the above theorem: we can identify a part of the nef cone for which the weak Lefschetz principle holds. Let $Y \subset X$ be a generic hyperplane section. If the dimension n of X is ≥ 4 , pull-back induces a natural isomorphism $N^1 X \cong N^1 Y$. Thus it makes sense to ask whether the nef cones $\text{Nef}(X)$ and $\text{Nef}(Y)$ coincide. The answer is negative in general, as shown by Hassett–Lin–Wang [15]. On the other hand, the answer is positive for certain Fano varieties ([26], [15], [18], [1], [6], [24]). Using the above Theorem, it turns out that the K_X -visible part cuts out a part where weak Lefschetz holds for the nef cone:

Corollary. (=Corollary 27) *Let X be a smooth projective complex variety of dimension $n \geq 4$, and let $Y \subset X$ be any ample hypersurface. Then*

$$\partial \text{Nef}(X)_{\text{visible}} \cap \text{int}(\text{Mob}(X)) \subset \partial \text{Nef}(Y) \cap \partial \text{Nef}(X) .$$

This is proven using a result of Demailly–Peternell–Schneider [10] (cf. also [20]), which says that a divisor restricting to an ample divisor on Y is 1-ample on X .

We prove a result similar to Theorem 19, by similar means, for the q -ample cone (where q may be > 1). This result is a bit more awkward to state. As a matter of notation, we introduce the cone $Bq\text{Amp}(X)$; this is defined as the cone of those \mathbb{R} -divisors which have augmented base locus of dimension $\leq q$.

Theorem. (=Theorem 31) *Let X be a smooth projective variety of dimension n . For any non-negative integer q , we have*

$$\partial \text{Nef}(X)_{\text{visible}} \cap B(n - 1 - q)\text{Amp}(X) \subset \overline{\partial q\text{Amp}(X)} .$$

Here is how this paper is organized. The first two sections are of a preliminary nature. The first concerns several cones of divisors related to the q -ample cones; the second contains some results about contractions that will be needed. Section 3 contains the proof of Theorem 19 and its corollaries. In section 4, we prove Theorem 31.

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Convention. *In this paper, all varieties will be (quasi-)projective algebraic varieties defined over the complex numbers.*

1. CONES

This section contains notation and basic results concerning several cones of divisors related to the q -ample cones. These cones have been introduced by Küronya [20] and de Fernex–Küronya–Lazarsfeld [13].

Definition 1. *Let X be a projective variety. A line bundle L on X is called q -ample if for every coherent sheaf \mathcal{F} there exists an integer m_0 such that*

$$H^i(X, \mathcal{F} \otimes L^{\otimes m}) \text{ for all } i > q \text{ and } m > m_0.$$

A \mathbb{Q} -Cartier divisor is called q -ample if some integral multiple is q -ample. An \mathbb{R} -Cartier divisor D is called q -ample if it can be written as a sum

$$D = cL + A ,$$

where $c \in \mathbb{R}_{>0}$, L is a q -ample line bundle and A is an ample \mathbb{R} -Cartier divisor. We will denote

$$q\text{Amp}(X) \subset N^1(X)$$

the cone generated by q -ample divisors.

Remark 2. The consistency of the definition for \mathbb{R} -divisors with the one for \mathbb{Q} -divisors is proven by Totaro [25, Theorem 8.3]. The cones $q\text{Amp}(X)$ are open cones [25, Theorem 8.3].

Theorem 3. ([25, Theorem 9.1]) Let X be a projective variety of dimension n . The cone $(n-1)\text{Amp}(X)$ is the complement in $N^1 X$ of the negative of the pseudo-effective cone of X .

Definition 4. Let X be a projective variety.

(i) An \mathbb{R} -divisor L on X is called B q -ample if the augmented base locus $B_+(L)$ has dimension $\leq q$. We will denote

$$Bq\text{Amp}(X) \subset N^1(X)$$

the cone generated by B q -ample divisors.

(ii) Let H_1, \dots, H_q be very ample divisors on X . An \mathbb{R} -divisor L on X is called (H_1, \dots, H_q) -ample if the restriction

$$L|_{h_1 \cap \dots \cap h_q}$$

is ample, for $h_i \in |H_i|$ generic. An \mathbb{R} -divisor is said to be H q -ample if it is (H_1, \dots, H_q) -ample, for certain very ample H_1, \dots, H_q . We will denote

$$Hq\text{Amp}(X) = \bigcup_{(H_1, \dots, H_q) \text{ very ample}} (H_1, \dots, H_q)\text{Amp}(X) \subset N^1(X)$$

the cone generated by H q -ample divisors.

Remark 5. The augmented base locus $B_+(L) \subset X$ is the locus where L fails to be ample; for the definition and properties, cf. [11] and [12].

Remark 6. It is easily seen that

$$B0\text{Amp}(X) = H0\text{Amp}(X) = \text{Amp}(X),$$

while $B(n-1)\text{Amp}(X) = \text{Big}(X)$. The cones $Bq\text{Amp}(X)$ are open [8, Theorem 4.5], and $B(n-2)\text{Amp}(X)$ coincides with the interior of the cone of mobile divisors:

$$B(n-2)\text{Amp}(X) = \text{Mob}(X) \setminus \partial \text{Mob}(X)$$

([9, Lemma 3.1]).

Remark 7. The cones $Bq\text{Amp}$ (or rather, their closure) have been studied by Payne [22] and Choi [8]. It is established by Choi [8, Theorem 4.5] that the closure of $Bq\text{Amp}(X)$ can be described in terms of the diminished base locus:

$$\overline{Bq\text{Amp}(X)} = \{L \in N^1 X \mid \dim B_-(L) \leq q\}.$$

Proposition 8. (Küronya [20]) Let X be a smooth projective variety. For any $0 \leq q \leq n-1$, there are inclusions of cones

$$Bq\text{Amp}(X) \subset Hq\text{Amp}(X) \subset q\text{Amp}(X).$$

Proof. For the first inclusion, it is easily seen that actually

$$Bq\text{Amp}(X) \subset \bigcap_{(H_1, \dots, H_q) \text{ very ample}} (H_1, \dots, H_q)\text{Amp}(X);$$

indeed, suppose L is such that $\dim B_+(L) \leq q$. For any H_1, \dots, H_q very ample and $h_i \in |H_i|$ generic, $B_+(L) \cap h_1 \cap \dots \cap h_q$ has dimension ≤ 0 . But

$$B_+(L|_{h_1 \cap \dots \cap h_q}) \subset B_+(L) \cap h_1 \cap \dots \cap h_q$$

[20,], so $L|_{h_1 \cap \dots \cap h_q}$ is ample. The second inclusion is a vanishing theorem proven by K ronya [20, Theorem 1.1]; this was also proven by Demailly–Peternell–Schneider [10, Theorem 3.4]. \square

Remark 9. Both inclusions in Proposition 8 may be strict. For the second inclusion, K ronya provides an example [20, Example 1.13] where

$$H(n-1)\text{Amp}(X) \neq (n-1)\text{Amp}(X).$$

For the first inclusion, let X be a surface. Then any line bundle L which is not big and such that $-L$ is not pseudo-effective is in

$$H1\text{Amp}(X) \setminus B1\text{Amp}(X).$$

A more subtle example is [20, Example 1.7], which exhibits a big line bundle L on a threefold X , satisfying

$$L \in H1\text{Amp}(X) \setminus B1\text{Amp}(X).$$

2. MMP

In this section, we collect some results about minimal model theory and contractions.

Definition 10. ([12]) A divisor L is called stable if $B_-(L)$ and $B_+(L)$ coincide.

Proposition 11. ([12, Proposition 1.29]) The stable divisors form an open and dense subset in N^1X .

Lemma 12. Let X be a smooth projective variety, and L on X an \mathbb{R} -divisor which is big and stable. Let

$$f: X \dashrightarrow X_{\min}$$

be an L -MMP, i.e. f_*L is nef. Let $\text{Exc}(f) \subset X$ denote the complement of the maximal open subset over which f is an isomorphism. Then

$$B_+(L) \supset \text{Exc}(f).$$

Proof. Let $E \subset \text{Exc}(f)$ be an irreducible component. Then, there is some index $0 < i < r$, such that $-(f_i)_*L$ is ψ_i -ample on the strict transform E_i of E in X_i . This implies

$$E_i \subset B_+((f_i)_*L)$$

(indeed, E_i is covered by curves on which $(f_i)_*L$ is negative, and such curves lie in the stable base locus of $(f_i)_*L$). But then, applying the following proposition to a resolution of indeterminacy of f_i , we see that E must lie in $B_+(L)$.

Proposition 13. (*Boucksom–Broustet–Pacienza [7, Proposition 1.5]*) *Let $\pi: \tilde{X} \rightarrow X$ be a birational morphism between normal projective varieties. Let F be an effective π -exceptional divisor. Then for any big \mathbb{R} -divisor L on X , we have*

$$B_+(\pi^*L + F) = \pi^{-1}(B_+(L)) \cup \text{Exc}(\pi) .$$

□

Remark 14. *With some more work, one can in fact prove that equality holds in Lemma 12; we don't need this in this paper.*

Theorem 15. (*Wiśniewski [26]*) *Let X be a smooth projective variety, and let*

$$\psi: X \rightarrow Z$$

be the contraction of a K_X -negative extremal ray. Suppose all fibres of ψ are of dimension ≤ 1 . Then Z is smooth, and ψ is either the blow-up of Z along a smooth codimension 2 subvariety, or a conic bundle over Z .

Proof. [26, Theorem 1.2] (cf. also [4, Theorem 4.1].

□

Theorem 16. (*Wiśniewski [26], Ionescu [17]*) *Let X be a smooth projective variety of dimension n , and let R be a K_X -negative extremal ray of length*

$$\ell(R) := \min\{-K_X \cdot C \mid C \text{ rational curve}, C \in R\} .$$

Let ψ be the contraction of R , and let E be an irreducible component of the locus of R . Let F be an irreducible component of a fiber of the restriction of ψ to E . Then

$$\dim E + \dim F \geq n + \ell(R) - 1 .$$

Proof. [26, Theorem 1.1] or [17, Theorem 0.4].

□

3. 1-AMPLE

This section is about the cone of 1-ample divisors. Here we prove Theorem 19 stated in the introduction.

Definition 17. *Let X be a projective variety. The K_X -visible part of $\partial\text{Nef}(X)$ is defined as*

$$\partial\text{Nef}(X)_{\text{visible}} := \{D \in \partial\text{Nef}(X) \mid \overline{K_X D} \cap \text{Nef}(X) = D\} .$$

Here $\overline{K_X D}$ denotes the line segment joining K_X to D .

Remark 18. *This notion is considered also in [19, Theorem 1]. The definition is interesting only when $K_X \notin \text{Nef}(X)$; if K_X is nef, the line segment $\overline{K_X D}$ contains more than one point and we have*

$$\partial\text{Nef}(X)_{\text{visible}} = \emptyset .$$

The other extreme is when X is Fano; then we have

$$\partial\text{Nef}(X)_{\text{visible}} = \partial\text{Nef}(X) .$$

Theorem 19. *Let X be a smooth projective variety.*

(i)

$$\partial\mathrm{Nef}(X)_{\mathrm{visible}} \cap \mathrm{int}(\mathrm{Mob}(X)) \subset \overline{\partial 1\mathrm{Amp}(X)}.$$

(ii) Suppose X is not the blow-up of a smooth projective variety Y along a smooth codimension 2 subvariety. Then

$$\partial\mathrm{Nef}(X)_{\mathrm{visible}} \cap \mathrm{Big}(X) \subset \overline{\partial 1\mathrm{Amp}(X)}.$$

(iii) Suppose X is not a conic bundle over a smooth projective variety Y , nor a blow-up of a smooth projective variety along a smooth codimension 2 subvariety. Then

$$\partial\mathrm{Nef}(X)_{\mathrm{visible}} \subset \overline{\partial 1\mathrm{Amp}(X)}.$$

Proof.

(i) We will prove the following:

Proposition 20. *Let $L = K_X + A$, where A is an ample \mathbb{R} -divisor. Suppose L is stable and*

$$L \in 1\mathrm{Amp}(X) \cap \mathrm{int}(\mathrm{Mob}(X)).$$

Then L is ample.

This suffices to prove Theorem 19(i). Indeed, suppose there is an element

$$D \in \partial\mathrm{Nef}(X)_{\mathrm{visible}} \cap \mathrm{int}(\mathrm{Mob}(X))$$

that is in the interior of $\overline{1\mathrm{Amp}(X)}$ (i.e. D is 1-ample). Then we can also find

$$D' \in \partial\mathrm{Nef}(X)_{\mathrm{visible}}^\circ \cap \mathrm{int}(\mathrm{Mob}(X))$$

that is 1-ample. Here $\partial\mathrm{Nef}(X)_{\mathrm{visible}}^\circ$ denotes the relative interior of $\partial\mathrm{Nef}(X)_{\mathrm{visible}}$. By definition of the K_X -visible part, D' is of the form $D' = m(K_X + A)$, for some ample \mathbb{R} -divisor A and $m \in \mathbb{R}$. Now, $\frac{1}{m}D' = K_X + A$ is also in

$$\partial\mathrm{Nef}(X)_{\mathrm{visible}} \cap \mathrm{int}(\mathrm{Mob}(X)) \cap 1\mathrm{Amp}(X).$$

What's more,

$$D'' = K_X + (1 - \epsilon)A \in \mathrm{int}(\mathrm{Mob}(X)) \cap 1\mathrm{Amp}(X)$$

for $0 < \epsilon$ small enough (since these are open cones). Since stable divisors are open and dense in N^1X , there exists $\epsilon > 0$ such that D'' is stable. Then Proposition 20 implies that D'' is ample, and hence D' is ample: contradiction.

So let's prove Proposition 20.

Since A is ample, there exists an effective \mathbb{R} -divisor Δ numerically equivalent to A and such that (X, Δ) is klt. According to [5, Theorem 1.2], there is an L -MMP

$$\phi: X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_{\min},$$

where ϕ_*L on X_{\min} is nef. Each step $\phi_i: X_i \dashrightarrow X_{i+1}$ in the program is the flip of a morphism

$$\psi_i: X_i \rightarrow Z_i,$$

where ψ_i is the (birational) contraction of an L -negative extremal ray. Since L is stable, the exceptional locus of ϕ is contained in $B_+(L)$ (Lemma 12), hence it is of dimension $\leq n - 2$

(where $n = \dim X$). That is, all the ψ_i in the program must be small contractions. Consider now the first of these small contractions

$$\psi = \psi_0: X \rightarrow Z_0.$$

Since $K_X < L$, ψ is the contraction of a K_X -negative extremal ray. If all fibres of ψ are of dimension ≤ 1 , the contraction ψ cannot be small by Theorem 15, so there must exist a fibre with an irreducible component F of dimension $f \geq 2$. Since $-L$ is ψ -ample, we have

$$-L|_F \in \text{Amp}(F) \subset \text{Big}(F).$$

Using Theorem 3, this implies

$$L|_F \notin (f-1)\text{Amp}(F).$$

But this leads to a contradiction: L is 1-ample, so the restriction to any subvariety must be 1-ample as well.

We find that ψ is the identity, so the MMP cannot get started and $X = X_{\min}$. That is, L must be nef. Since L is stable, $B_+(L) = B_-(L) = \emptyset$ and L is ample.

(ii) In analogous fashion to the proof of (i), it will suffice to prove:

Proposition 21. *Let X be as in Theorem 19(ii), and let $L = K_X + A$, where A is an ample \mathbb{R} -divisor. Suppose L is stable and*

$$L \in 1\text{Amp}(X) \cap \text{Big}(X).$$

Then L is ample.

To prove the proposition, consider again an L -MMP (which exists thanks to [5,]). Let

$$\psi: X \rightarrow Z$$

be the first contraction of the program. Since L is big, the contraction ψ is birational. Just as above, we find that ψ cannot be small, so ψ must be a divisorial contraction. If all fibres of ψ have dimension ≤ 1 , ψ is a blow-up of a smooth projective Y with smooth center of codimension 2 (Theorem 15); this is excluded by hypothesis. So there must be a fibre with an irreducible component F of dimension ≥ 2 , which again contradicts the fact that $L|_F$ is 1-ample.

(iii) It will suffice to prove the following statement:

Proposition 22. *Let X be as in Theorem 19(iii), and let $L = K_X + A$, where A is an ample \mathbb{R} -divisor. Suppose L is stable and 1-ample. Then L is nef.*

We first remark that in case L is big, Proposition 22 follows from Proposition 21. In case L is pseudo-effective, L is a limit of big divisors which are stable and 1-ample, and it follows from Proposition 21 that L is nef. Suppose L is not pseudo-effective. According to [5, Corollary 1.3.2], there exists an L -MMP such that on X_{\min} there is a Mori fibre space structure, i.e. a morphism

$$g: X_{\min} \rightarrow Y$$

such that $-\phi_*L$ is g -ample. Just as in case (ii), we find there can be no birational contraction in the program, so we have $X = X_{\min}$. If the Mori fibre space has only fibres of dimension 1, it is

a conic bundle over a smooth Y (Theorem 15), which is excluded by hypothesis. So there exists a fibre of g with an irreducible component F of dimension ≥ 2 ; this contradicts the fact that

$$L|_F \in 1\text{Amp}(F) .$$

□

Corollary 23. *Let X be a smooth projective variety, and suppose K_X is 1-ample. Then*

$$\partial\text{Nef}(X)_{\text{visible}} \subset \overline{\partial\text{Mob}(X)} .$$

Proof. It suffices to prove that the relative interior $\partial\text{Nef}(X)_{\text{visible}}^\circ$ is in the boundary of the mobile cone. But if K_X is 1-ample, every L on $\partial\text{Nef}(X)_{\text{visible}}^\circ$ is also 1-ample (since L is a sum of ample plus 1-ample). But then Theorem 19(i) implies that L cannot live in the interior of $\text{Mob}(X)$. □

Corollary 24. *Let X be a smooth projective variety, and suppose*

$$\partial\text{Nef}(X)_{\text{visible}} \cap \text{int}(\text{Mob}(X)) \neq \emptyset .$$

Then $1\text{Amp}(X)$ is a strictly convex cone.

Proof. The hypothesis implies that the dimension of X is at least 3. In case the Picard number of X is 1, the statement is clear from Theorem 3. Suppose the Picard number is 2. The cone $\overline{1\text{Amp}(X)}$ has 2 extremal rays, and by Theorem 19(i) one of them is also an extremal ray of $\text{Nef}(X)$. On the other hand, $\overline{1\text{Amp}(X)}$ lies outside of $-\text{Amp}(X)$ (Theorem 3), so $\overline{1\text{Amp}(X)}$ must be convex.

The argument for Picard number ≥ 3 is similar: in this case, we have

$$\dim \partial\text{Nef}(X)_{\text{visible}} \geq 2 ,$$

which means that $\partial\text{Nef}(X)_{\text{visible}}$ contains infinitely many rays. Since the visible part is locally rationally polyhedral (this is the cone theorem, stated in this form in [19, Theorem 1]), there exists a ray

$$R \in \partial\text{Nef}(X)_{\text{visible}}$$

which lies in the relative interior of a face F of $\text{Nef}(X)$. Let $h \subset N^1X_{\mathbb{R}}$ denote the unique hyperplane containing F ; the claim is now that $1\text{Amp}(X)$ lies on one side of h . To see this, suppose (by contradiction) there exists a divisor $D \in 1\text{Amp}(X)$ which lies on the “non-ample” side of h . Let $h_2 \subset N^1X_{\mathbb{R}}$ denote the 2-plane spanned by R and D . We find that any divisor $L \in R$ can be written

$$L = mD + A ,$$

for some $m \in \mathbb{R}_{>0}$ and A ample (this is most easily seen by restricting attention to the 2-plane h_2 : by construction, h_2 meets $\text{Amp}(X)$, and D lies outside of $-\text{Amp}(X) \cap h_2$, again by Theorem 3). But then L is 1-ample, contradicting Theorem 19(i). □

Corollary 25. (“almost Fano”) *Let X be a smooth projective complex variety, and suppose that either (1) $-K_X$ is ample, or (2) $-K_X$ is $\neq 0$ and nef and $\dim N^1X \geq 3$. Then:*

(i)

$$\partial \text{Nef}(X) \cap \text{int}(\text{Mob}(X))$$

is in the boundary of $\overline{1\text{Amp}(X)}$.

(ii) Suppose X is not the blow-up of a smooth projective variety Y along a smooth codimension 2 subvariety. Then

$$\partial \text{Nef}(X) \cap \text{Big}(X) \subset \overline{\partial 1\text{Amp}(X)} .$$

(iii) Suppose X is not a conic bundle over a smooth projective variety Y , nor a blow-up of a smooth projective variety along a smooth codimension 2 subvariety. Then

$$\text{Amp}(X) = 1\text{Amp}(X) .$$

Proof.

(i) If $-K_X$ is ample, clearly

$$\partial \text{Nef}(X)_{\text{visible}} = \partial \text{Nef}(X)$$

and we are done. Suppose now

$$-K_X \in \partial \text{Nef}(X) \setminus \{0\} .$$

Then we have

$$\partial \text{Nef}(X)_{\text{visible}} = \partial \text{Nef}(X) \setminus k ,$$

where k denotes the ray generated by $-K_X$. Applying Theorem 19(i), we find an inclusion

$$\left(\partial \text{Nef}(X) \setminus k \right) \cap \text{int}(\text{Mob}(X)) \subset \overline{\partial 1\text{Amp}(X)} .$$

Suppose (i) is not true, i.e.

$$k \subset \text{int}(\text{Mob}(X)) \cap 1\text{Amp}(X) .$$

Then, since 1Amp is an open cone,

$$D := -K_X - \epsilon A \in 1\text{Amp}(X)$$

for any ample A and ϵ sufficiently small. On the other hand, D lies outside the closed cone $\text{Nef}(X)$. Let's pick an ample \mathbb{R} -divisor A' close to A , but outside the plane spanned by A and k (this is possible if the ample cone has dimension ≥ 3). Then the line segment connecting A' to D crosses

$$\left(\partial \text{Nef}(X) \setminus k \right) \cap \text{int}(\text{Mob}(X)) ;$$

let's call the point of intersection B . The \mathbb{R} -divisor B is a sum of ample and 1-ample, hence B is 1-ample [25, Theorem 8.3]. On the other hand, B lies in the boundary of $\overline{1\text{Amp}(X)}$ and the 1-ample cone is open, so B cannot be 1-ample: contradiction.

(ii) and (iii) Similar. □

Remark 26. Suppose X is Fano, i.e. $-K_X$ is ample. The pseudo-index of X is defined as

$$\tau(X) = \min\{-K_X \cdot C \mid C \subset X \text{ rational curve}\} .$$

If $\tau(X)$ is ≥ 2 (respectively ≥ 3), the hypothesis of Corollary 25(ii) (respectively (iii)) is satisfied (this follows from Theorem 16). In this way, we recover [21, Proposition 29] as a special case of Corollary 25.

Corollary 27. (*"weak Lefschetz"*) *Let X be a smooth projective complex variety of dimension $n \geq 3$, and let $Y \subset X$ be a generic hyperplane section.*

(i)

$$\partial \text{Nef}(X)_{\text{visible}} \cap \text{int}(\text{Mob}(X)) \subset \partial \text{Nef}(Y) \cap \partial \text{Nef}(X) .$$

(ii) *Suppose X is not the blow-up of a smooth projective variety Y along a codimension 2 smooth subvariety. Then*

$$\partial \text{Nef}(X)_{\text{visible}} \cap \text{Big}(X) \subset \partial \text{Nef}(Y) \cap \partial \text{Nef}(X) .$$

(iii) *Suppose X is not a conic bundle over a smooth projective variety, nor a blow-up of a smooth projective variety along a smooth codimension 2 subvariety. Then*

$$\partial \text{Nef}(X)_{\text{visible}} \subset \partial \text{Nef}(Y) \cap \partial \text{Nef}(X) .$$

The following is an alternative formulation of Corollary 27(i). The reformulation of points (ii) and (iii) is left to the diligent reader.

Corollary 28. (*"ampleness criterion"*) *Let X be a smooth projective variety of dimension $n \geq 3$, and let L on X be a divisor of the form $L = K_X + A$, with A an ample \mathbb{R} -divisor. Suppose $L \in \text{int}(\text{Mob}(X))$. Then L is ample if and only if $L|_Y$ is ample for some generic hyperplane $Y \subset X$.*

Combining Corollaries 25 and 27, we get in particular:

Corollary 29. (*"weak Lefschetz for almost Fano"*) *Let X be a smooth projective complex variety of dimension $n \geq 3$. Suppose either (1) $-K_X$ is ample, or (2) $-K_X$ is nef and $\neq 0$ and $\dim N^1 X \geq 3$. Let $Y \subset X$ be a very ample divisor, generic in its linear system.*

(i)

$$\partial \text{Nef}(X) \cap \text{int}(\text{Mob}(X)) \subset \partial \text{Nef}(Y) \cap \partial \text{Nef}(X) .$$

(ii) *Suppose X is not the blow-up of a smooth variety along a smooth codimension 2 subvariety. Then*

$$\partial \text{Nef}(X) \cap \text{Big}(X) \subset \partial \text{Nef}(Y) \cap \partial \text{Nef}(X) .$$

(iii) *Suppose X is not a conic bundle over a smooth projective variety, nor a blow-up of a smooth projective variety along a smooth codimension 2 subvariety. Then*

$$\partial \text{Nef}(X) \subset \partial \text{Nef}(Y) .$$

(iv) *Let X be as in (iii) and $n \geq 4$. Then restriction induces an isomorphism*

$$\text{Nef}(X) \cong \text{Nef}(Y) .$$

Remark 30. *The statement of Corollary 29(iv) for X Fano was originally proven by Wiśniewski [26, p. 147 Corollary]. This provided the starting-block for much further work concerning weak Lefschetz for the ample cone ([15], [18], [1], [2], [6], [24]).*

4. q -AMPLE

This section is about the cone of q -ample divisors. We prove the result stated in the introduction:

Theorem 31. *Let X be a smooth projective variety of dimension n . For any non-negative integer q , we have*

$$\partial \text{Nef}(X)_{\text{visible}} \cap B(n-1-q)\text{Amp}(X) \subset \overline{\partial q \text{Amp}(X)}.$$

We actually prove a more general statement:

Theorem 32. *Let X be a smooth projective variety of dimension n , and define*

$$\tau = \min\{\ell(R) \mid R \text{ is a } K_X\text{-negative extremal ray}\}.$$

(i) *For any non-negative integer q such that $q \geq \tau - 2$, we have*

$$\partial \text{Nef}(X)_{\text{visible}} \cap B(n+\tau-q-2)\text{Amp}(X) \subset \overline{\partial q \text{Amp}(X)}.$$

(ii) *Suppose X is not the blow-up of a smooth variety Y along a smooth subvariety of codimension ≥ 2 . Then*

$$\partial \text{Nef}(X)_{\text{visible}} \cap \text{Big}(X) \subset \overline{\partial(\tau) \text{Amp}(X)}.$$

Proof.

(i) As in the proof of Theorem 19, one can restrict attention to the relative interior $\partial \text{Nef}(X)_{\text{visible}}^\circ$ and hence it suffices to prove the following:

Proposition 33. *Let L be a divisor of the form $L = K_X + A$, with A an ample \mathbb{R} -divisor. Suppose L is stable and*

$$L \in B(n+\tau-q-2)\text{Amp}(X) \cap q\text{Amp}(X).$$

Then L is ample.

To prove the proposition, consider an L -MMP

$$\phi: X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_{\min},$$

where either ϕ_*L is semi-ample on X_{\min} (if L is big), or there exists a Mori fibre space structure on X_{\min} (if L is not pseudo-effective). This exists thanks to [5]. Let

$$\psi: X \rightarrow Z$$

denote the first contraction of the L -MMP, let $V \subset X$ denote the exceptional locus of ψ and let F be a general fibre of $\psi|_V$. Note that $K_X < L$ so that ψ corresponds to the contraction of a K_X -negative extremal ray and Wiśniewski's theorem (Theorem 16) applies. This gives

$$\dim V + \dim F \geq n + \tau - 1.$$

Since $V \subset B_+(L)$ (Lemma 12), its dimension is $\leq n + \tau - q - 2$. It follows that

$$\dim F \geq q + 1.$$

By construction, $-L$ is ψ -ample, hence

$$L|_F \in -\text{Amp}(F) .$$

On the other hand the restriction of L to any subvariety is q -ample, so in particular

$$L|_F \in q\text{Amp}(F) .$$

But this is not possible if $\dim F \geq q + 1$:

$$q\text{Amp}(F) \subset (\dim F - 1)\text{Amp}(F) ,$$

and the cone $(\dim F - 1)\text{Amp}(F)$ is the complement of $-\text{Psef}(F)$: contradiction.

Since the L -MMP cannot get started, it is trivial. That is, either L is nef on X , or there exists a contraction of fibre type

$$g: X \rightarrow Z$$

which is L -negative and K_X -negative. The second possibility can be excluded, again using Wiśniewski's theorem: if F is a general fibre of g , we have

$$n + \dim F \geq n + \tau - 1 ,$$

i.e. there is a fibre F of dimension $\geq \tau - 1$. But supposing there is a fibre type contraction, L is not big which is only possible if $q = \tau - 2$. So L and the restriction $L|_F$ are $(\tau - 2)$ -ample, which contradicts the fact that

$$L|_F \in -\text{Amp}(F) \subset -\text{Big}(F) .$$

(ii) This follows once we have proven the following:

Proposition 34. *Let X be as in Theorem 32(ii), and let L be a divisor of the form $L = K_X + A$, with A an ample \mathbb{R} -divisor. Suppose L is big and τ -ample. Then L is ample.*

To prove the proposition, we apply [5,] to get an L -MMP

$$\phi: X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_{\min} ,$$

where ϕ_*L on X_{\min} is nef. Consider the first contraction

$$\psi: X \rightarrow Z$$

in this L -MMP. As above, let $V \subset X$ denote the exceptional locus of ψ and let F be a general fibre of $\psi|_V$. Note that $K_X < L$ so that ψ corresponds to the contraction of a K_X -negative extremal ray and hence Wiśniewski's theorem (Theorem 16) applies to ψ . If ψ is a small contraction (i.e. $\dim V \leq n - 2$), Wiśniewski's theorem gives

$$\dim F \geq \tau + 1 ,$$

and we get a contradiction with the fact that $L|_F$ is τ -ample. So ψ must be a divisorial contraction, and all fibres of $\psi|_V$ must be of dimension equal to τ (by Wiśniewski's theorem, each fibre has dimension $\geq \tau$, while the fact that L is τ -ample implies that each fibre has dimension $\leq \tau$). In this case, a result of Andreata–Occhetta [3, Theorem 5.1] informs us that ψ identifies X with a blow-up of some smooth projective variety Y along a smooth subvariety; this is excluded by hypothesis.

Altogether, we find there can be no contraction and hence $X = X_{\min}$ and L is already nef. It remains to prove ampleness of L . To this end, note that

$$L' = K_X + (1 - \epsilon)A$$

is still big and $(\tau - 1)$ -ample for ϵ sufficiently small (since $\text{Big}(X)$ and $(\tau - 1)\text{Amp}(X)$ are open cones). Applying the above reasoning to L' , we find that L' is nef. But then

$$L = L' + \epsilon A$$

is ample. □

Corollary 35. (*"weak Lefschetz"*) *Let X and τ be as in Theorem 32.*

(i) *Let $Y \subset X$ be a generic complete intersection of codimension $q \leq n - 2$. Then*

$$\partial\text{Nef}(X)_{\text{visible}} \cap B(n + \tau - q - 2)\text{Amp}(X) \subset \partial\text{Nef}(Y) \cap \partial\text{Nef}(X) .$$

(ii) *Suppose X is not the blow-up of a smooth variety along a smooth subvariety of codimension ≥ 2 . Let $Y \subset X$ be a generic complete intersection of codimension τ . Then*

$$\partial\text{Nef}(X)_{\text{visible}} \cap \text{Big}(X) \subset \partial\text{Nef}(Y) \cap \partial\text{Nef}(X) .$$

Proof. This is immediate from Theorem 31, once one knows that Hq -ample implies q -ample (Proposition 8). □

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